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The multipliers between the mixed norm spaces in C^n [☆]

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Abstract

In this paper we discuss the pointwise multipliers between the mixed norm spaces on the unit ball of C^n . For normal functions φ_1 and φ_2 , we give several full characterizations of the pointwise multipliers $M(H(p, q, \varphi_1), H(p, q, \varphi_2))$ for $0 < p \leq \infty$, $0 < q \leq \infty$ and $M(H(p, p, \varphi_1), H(q, q, \varphi_2))$ for $0 < p < \infty$, $0 < q < \infty$.

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1. Introduction

Let dv be the Lebesgue measure on the unit ball B normalized so that $v(B) = 1$, and $d\sigma$ be the normalized rotation invariant measure on the boundary ∂B of B so that $\sigma(\partial B) = 1$.

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The class of all holomorphic functions on B is denoted by $H(B)$ and H^∞ denotes the class of all bounded holomorphic functions on B .

A positive continuous function φ on $[0, 1)$ is normal, if there are constants $0 < a < b$ such that

- (i) $\frac{\varphi(r)}{(1-r)^a}$ is decreasing for $0 \leq r < 1$ and $\lim_{r \rightarrow 1^-} \frac{\varphi(r)}{(1-r)^a} = 0$;
(ii) $\frac{\varphi(r)}{(1-r)^b}$ is increasing for $0 \leq r < 1$ and $\lim_{r \rightarrow 1^-} \frac{\varphi(r)}{(1-r)^b} = \infty$.

Let $0 < p \leq \infty$, $0 < q \leq \infty$ and φ be normal on $[0, 1)$. f is said to belong to the mixed norm space $L(p, q, \varphi)$ if f is measurable function on B and $\|f\|_{p,q,\varphi} < \infty$, where

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 r^{2n-1} (1-r)^{-1} \varphi^p(r) M_q^p(r, f) dr \right\}^{\frac{1}{p}} \quad (0 < p < \infty, 0 < q \leq \infty),$$

$$\|f\|_{\infty,q,\varphi} = \sup_{0 \leq r < 1} \varphi(r) M_q(r, f), \quad M_\infty(r, f) = \sup_{\xi \in \partial B} |f(r\xi)|,$$

$$M_q(r, f) = \left\{ \int_{\partial B} |f(r\xi)|^q d\sigma(\xi) \right\}^{\frac{1}{q}} \quad (0 < q < \infty).$$

If $0 < p = q < \infty$, then $L(p, q, \varphi)$ is just the space

$$L^p(\varphi) = \left\{ f \text{ is measurable function on } B: \int_B |f(z)|^p \frac{\varphi^p(|z|)}{1-|z|} dv(z) < \infty \right\}.$$

Let $H(p, q, \varphi) = L(p, q, \varphi) \cap H(B)$. If $0 < p = q < \infty$, then $H(p, q, \varphi)$ is just the weighted Bergman space $L_a^p(\varphi)$. In particular, $H(p, q, \varphi)$ is Bergman space L_a^p if $0 < p = q < \infty$ and $\varphi(r) = (1-r)^{1/p}$. Otherwise, if $p = q = 2$ and $\varphi(r) = (1-r)^{-\beta/2}$ ($\beta < 0$), then $H(p, q, \varphi)$ is Dirichlet type space D_β (see Lemma 2.1 in [1]).

Let X and Y be two spaces of holomorphic function on B . We call g a pointwise multiplier from X to Y if $gf \in Y$ for every $f \in X$. The collection of all pointwise multipliers from X to Y is denoted $M(X, Y)$.

The multiplier theory has been studied for a long time, and a lot of results have been obtained (for example, [2–8]). All of these results are very useful in the discussion of the theory of operator and the other properties of related function spaces. In this paper, we will try to characterize the pointwise multipliers from the mixed norm spaces $H(p, q, \varphi_1)$ to $H(p, q, \varphi_2)$ ($0 < p \leq \infty$, $0 < q \leq \infty$). At the same time, we will give the corresponding results from spaces $H(p, p, \varphi_1)$ to $H(q, q, \varphi_2)$ ($0 < p < \infty$, $0 < q < \infty$) also. Our main results are the following:

Theorem A. Suppose $0 < p \leq \infty$, $0 < q \leq \infty$, φ_1 and φ_2 are normal on $[0, 1)$. Then $g \in M(H(p, q, \varphi_1), H(p, q, \varphi_2))$ if and only if $g \in H(B)$ and

$$\sup_{z \in B} \left\{ \frac{\varphi_2(|z|)}{\varphi_1(|z|)} |g(z)| \right\} < \infty. \quad (1.1)$$

Theorem B. Suppose $0 < p \leq q < \infty$, φ_1 and φ_2 are normal on $[0, 1)$. Then $g \in M(H(p, p, \varphi_1), H(q, q, \varphi_2))$ if and only if $g \in H(B)$ and

$$\sup_{z \in B} \left\{ (1 - |z|^2)^{(\frac{1}{q} - \frac{1}{p})n} \frac{\varphi_2(|z|)}{\varphi_1(|z|)} |g(z)| \right\} < \infty. \quad (1.2)$$

Theorem C. Suppose $0 < q < p < \infty$, φ_1, φ_2 and φ_2/φ_1 are normal on $[0, 1)$. Then $g \in M(H(p, p, \varphi_1), H(q, q, \varphi_2))$ if and only if $g \in H(k, k, \varphi_2/\varphi_1)$, where $1/k = 1/q - 1/p$.

2. Some lemmas

In the following, $z = (z_1, \dots, z_n)$, $w = (w_1, \dots, w_n)$, $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$. We will use the symbol c or c_1 to denote a positive constant which does not depend on variables z, w and may depend on some parameters, not necessarily the same at each occurrence.

Let $E^*(w, r) = \{z \in B: |\varphi_w(z)| < r\}$ ($w \in B$, $0 < r < 1$), where φ_w is the Möbius transformation of B with $\varphi_w(0) = w$, $\varphi_w(w) = 0$, $\varphi_w = \varphi_w^{-1}$. If $z \in E^*(w, r)$, then

$$\frac{1-r}{1+r}(1-|w|) \leq 1-|z| \leq \frac{1+r}{1-r}(1-|w|), \quad (2.1)$$

$$\frac{1-r}{1+r}(1-|w|^2) \leq 1-|z|^2 \leq \frac{1+r}{1-r}(1-|w|^2), \quad (2.2)$$

$$\sqrt{\frac{1-r}{1+r}}(1-|w|^2) \leq |1 - \langle z, w \rangle| \leq \frac{1}{1-r}(1-|w|^2). \quad (2.3)$$

In fact, if $z \in E^*(w, r)$, then

$$\begin{aligned} \frac{(1-|w|^2)(1-|z|^2)}{(1-|z||w|)^2} &\geq \frac{(1-|w|^2)(1-|z|^2)}{|1 - \langle z, w \rangle|^2} = 1 - |\varphi_w(z)|^2 > 1 - r^2 \\ \Rightarrow (1-r^2|w|^2)|z|^2 - 2(1-r^2)|w||z| + |w|^2 - r^2 &< 0 \\ \Rightarrow |z| < (|w| + r)/(1+r|w|) \quad \text{and} \quad |w| < (|z| + r)/(1+r|z|). \end{aligned}$$

Thus, (2.1)–(2.3) can be obtained easily.

Otherwise, if φ is normal on $[0, 1)$ and $z \in E^*(w, r)$, then by (2.1) and the definition of normal function, we have

$$\left\{ \frac{1-r}{1+r} \right\}^b \varphi(|w|) \leq \varphi(|z|) \leq \left\{ \frac{1+r}{1-r} \right\}^b \varphi(|w|). \quad (2.4)$$

Lemma 2.1. Suppose $0 < q \leq \infty$, and $f \in H(B)$, $w \in B$. Then

$$|f(w)| \leq \frac{\{4n \cdot 36^n\}^{1/q}}{(1-|w|^2)^{n/q}} M_q \left(\frac{|w|+1}{2}, f \right).$$

Proof. If $q = \infty$, the result can be obtained by the maximum modulus principle.

Now let $0 < q < \infty$. If $z \in E^*(w, r)$, then $|z| < (|w| + r)/(1 + r|w|)$. This implies that $E^*(w, r) \subset \frac{|w|+1}{2}B = \{z: |z| < \frac{1+|w|}{2}\}$ when $0 < r < 1/(2 + |w|)$. In particular, $E^*(w, \frac{1}{3}) \subset \frac{|w|+1}{2}B$. By subharmonicity of $|f|^q$ and the monotonicity of $M(t, f)$ we have

$$\begin{aligned}
|f(w)|^q &= |f \circ \varphi_w(0)|^q \leq 9^n \int_{E^*(0, \frac{1}{3})} |f \circ \varphi_w(z)|^q dv(z) \\
&= 9^n \int_{E^*(w, \frac{1}{3})} |f(\eta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \eta, w \rangle|^2} \right\}^{n+1} dv(\eta) \\
&\leq 9^n \int_{\frac{1+|w|}{2} B} |f(\eta)|^q \left\{ \frac{1 - |w|^2}{|1 - \langle \eta, w \rangle|^2} \right\}^{n+1} dv(\eta) \\
&\leq 2n9^n (1 - |w|^2)^{n+1} \int_0^{\frac{1+|w|}{2}} \frac{1}{(1 - t|w|)^{2n+2}} dt M_q^q \left(\frac{|w| + 1}{2}, f \right) \\
&= 2n9^n (1 - |w|^2)^{n+1} \int_{\frac{(1-|w|)(2+|w|)}{2}}^1 \frac{1}{|w|t^{2n+2}} dt M_q^q \left(\frac{|w| + 1}{2}, f \right) \\
&\leq 2n9^n (1 - |w|^2)^{n+1} \int_{1-|w|}^1 \frac{1}{|w|t^{2n+2}} dt M_q^q \left(\frac{|w| + 1}{2}, f \right) \\
&\leq \frac{4n \cdot 36^n}{(2n+1)(1 - |w|^2)^n} M_q^q \left(\frac{|w| + 1}{2}, f \right) \sup_{0 < x < 1} \frac{1 - (1-x)^{2n+1}}{x} \\
&= \frac{4n \cdot 36^n}{(1 - |w|^2)^n} M_q^q \left(\frac{|w| + 1}{2}, f \right). \quad \square
\end{aligned}$$

Lemma 2.2. Suppose $0 < p < \infty$ and φ is normal on $[0, 1)$. If $f \in H(p, p, \varphi)$, then

$$|f(z)| \leq \frac{\{2n4^n \cdot 3^{n+2+pb}\}^{1/p}}{\varphi(|z|)(1 - |z|^2)^{n/p}} \|f\|_{p,p,\varphi} \quad (z \in B).$$

Proof. For $0 < r < 1$ and $z \in B$, by the subharmonicity of $|f(z)|^p$ and (2.3) we have

$$\begin{aligned}
|f(z)|^p &\leq \frac{1}{r^{2n}} \int_{E^*(z,r)} |f(w)|^p \left\{ \frac{1 - |z|^2}{|1 - \langle w, z \rangle|^2} \right\}^{n+1} dv(w) \\
&\leq \frac{1}{r^{2n}} \left(\frac{1+r}{1-r} \right)^{n+1} \frac{1}{(1 - |z|^2)^{n+1}} \int_{E^*(z,r)} |f(w)|^p dv(w). \quad (2.5)
\end{aligned}$$

In particular, given $r = 1/2$. By (2.1), (2.4) and (2.5) we have

$$|f(z)|^p \leq \frac{4^n \cdot 3^{n+2+pb}}{(1 - |z|^2)^n \varphi^p(|z|)} \int_{E^*(z, \frac{1}{2})} (1 - |w|)^{-1} \varphi^p(|w|) |f(w)|^p dv(w)$$

$$\begin{aligned}
&\leq \frac{4^n \cdot 3^{n+2+pb}}{(1-|z|^2)^n \varphi^p(|z|)} \int_B (1-|w|)^{-1} \varphi^p(|w|) |f(w)|^p dv(w) \\
&\leq \frac{2n4^n \cdot 3^{n+2+pb}}{\varphi^p(|z|)(1-|z|^2)^n} \|f\|_{p,p,\varphi}^p. \quad \square
\end{aligned}$$

Let $\beta(\cdot, \cdot)$ denote the Bergman metric on B . The Bergman ball $E(z, r)$ with center $z \in B$ and radius $r > 0$ is defined as $E(z, r) = \{w \in B: \beta(z, w) < r\}$. It is known that

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad E^*(z, \tanh r) = E(z, r) \quad (0 < r < \infty).$$

Lemma 2.3. Suppose μ is a positive Borel measure on B , and φ is normal on $[0, 1)$. Let $0 < q < p < \infty$ and $r_0 = (e - e^{-1})/(e + e^{-1})$. Then a sufficient and necessary condition for there to exist a constant c such that

$$\left\{ \int_B |f(z)|^q d\mu(z) \right\}^{\frac{1}{q}} \leq c \left\{ \int_B |f(z)|^p (1-|z|)^{-1} \varphi^p(|z|) dv(z) \right\}^{\frac{1}{p}}$$

for all $f \in H(p, p, \varphi)$ is that $\int_B \hat{\mu}(|z|)^s (1-|z|)^{-1} \varphi^p(|z|) dv(z) < \infty$, where

$$\frac{1}{s} + \frac{q}{p} = 1, \quad \hat{\mu}(|z|) = \frac{\mu(E^*(z, r_0))}{(1-|z|^2)^n \varphi^p(|z|)}.$$

Proof. Taking $w^*(z) = (1-|z|)^{-1} \varphi^p(|z|)$ in Theorem 4 in [9]. \square

3. Proofs of main results

Proof of Theorem A. The sufficient condition is obvious, and the details are omitted there. We prove the necessary condition only.

If $g \in M(H(p, q, \varphi_1), H(p, q, \varphi_2))$, then $C_g: f \rightarrow gf$ is a bounded linear operator from $H(p, q, \varphi_1)$ to $H(p, q, \varphi_2)$ by the closed graph theorem. On the other hand, we have $g = gf \in H(p, q, \varphi_2) \subset H(B)$ by taking $f(z) = 1 \in H(p, q, \varphi_1)$. Let a_i, b_i ($i = 1, 2$) be the parameters on the definition of φ_1, φ_2 and $a = \min\{a_1, a_2\}$, $b = \max\{b_1, b_2\}$. For any $w \in B$, we take

$$f_w(z) = \frac{(1-|w|^2)^{b+1}}{\varphi_1(|w|)(1-\langle z, w \rangle)^{n/q+b+1}}.$$

By Proposition 1.4.10 in [10] we have

$$M_q(r, f_w) \leq \frac{c(1-|w|^2)^{b+1}}{\varphi_1(|w|)(1-r|w|)^{b+1}}. \quad (3.1)$$

If $0 < p < \infty$, by the definition of normal function, Lemma 2.2 in [11] and (3.1) we get

$$\begin{aligned}
\|f_w\|_{p,q,\varphi_1}^p &= \int_0^1 r^{2n-1} (1-r)^{-1} \varphi_1^p(r) M_q^p(r, f_w) dr \\
&\leq c \int_0^1 (1-r)^{-1} \varphi_1^p(r) \frac{(1-|w|^2)^{p(b+1)}}{\varphi_1^p(|w|)(1-r|w|)^{p(b+1)}} dr \\
&= c \left(\int_0^{|w|} + \int_{|w|}^1 \right) (1-r)^{-1} \varphi_1^p(r) \frac{(1-|w|^2)^{p(b+1)}}{\varphi_1^p(|w|)(1-r|w|)^{p(b+1)}} dr \\
&\leq 2^{p(b+1)} c (1-|w|)^p \int_0^{|w|} \frac{(1-r)^{pb-1}}{(1-r|w|)^{p(b+1)}} dr \\
&\quad + 2^{p(b+1)} c (1-|w|)^{p(b+1)-pa} \int_{|w|}^1 \frac{(1-r)^{pa-1}}{(1-r|w|)^{p(b+1)}} dr \\
&\leq 2^{p(b+1)} c (1-|w|)^p \int_0^1 \frac{(1-r)^{pb-1}}{(1-r|w|)^{p(b+1)}} dr \\
&\quad + 2^{p(b+1)} c (1-|w|)^{p(b+1)-pa} \int_0^1 \frac{(1-r)^{pa-1}}{(1-r|w|)^{p(b+1)}} dr \\
&\leq c_1^p.
\end{aligned}$$

If $p = \infty$, by (3.1) and computation we have

$$\begin{aligned}
\sup_{0 \leq r < 1} M_q(r, f_w) \varphi_1(r) &\leq \left(\sup_{0 \leq r \leq |w|} + \sup_{|w| < r < 1} \right) \frac{c(1-|w|^2)^{b+1} \varphi_1(r)}{\varphi_1(|w|)(1-r|w|)^{b+1}} \\
&\leq c \left\{ \sup_{0 \leq r \leq |w|} \frac{(1-|w|^2)^{b+1} (1-r)^b}{(1-r|w|)^{b+1} (1-|w|)^b} + \sup_{|w| < r < 1} \frac{(1-|w|^2)^{b+1} (1-r)^a}{(1-r|w|)^{b+1} (1-|w|)^a} \right\} \leq c_1.
\end{aligned}$$

This shows that $\|f_w\|_{p,q,\varphi_1} \leq c_1$.

If $0 < p < \infty$, by monotonicity of $M_q(r, f)$ we have

$$\begin{aligned}
c_1 \|C_g\| &\geq \|C_g\| \|f_w\|_{p,q,\varphi_1} \geq \|gf_w\|_{p,q,\varphi_2} \\
&= \left\{ \int_0^1 r^{2n-1} (1-r)^{-1} \varphi_2^p(r) M_q^p(r, gf_w) dr \right\}^{\frac{1}{p}} \\
&\geq \left\{ \int_{\frac{1+|w|}{2}}^{\frac{3+|w|}{4}} r^{2n-1} (1-r)^{-1} \varphi_2^p(r) M_q^p(r, gf_w) dr \right\}^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
&\geq \left(\frac{1}{2}\right)^{\frac{2n-1}{p}} \left\{ \int_{\frac{1+|w|}{2}}^{\frac{3+|w|}{4}} (1-r)^{-1} \varphi_2^p(r) M_q^p(r, g f_w) dr \right\}^{\frac{1}{p}} \\
&\geq \left(\frac{1}{2}\right)^{\frac{2n-1}{p}} M_q\left(\frac{1+|w|}{2}, g f_w\right) \left\{ \int_{\frac{1+|w|}{2}}^{\frac{3+|w|}{4}} (1-r)^{-1} \varphi_2^p(r) dr \right\}^{\frac{1}{p}}. \quad (3.2)
\end{aligned}$$

Since

$$\frac{\varphi_2(|w|)}{(1-|w|)^b} (1-r)^b \leq \varphi_2(r) \quad \text{as } |w| < r < 1. \quad (3.3)$$

By (3.2), (3.3) and Lemma 2.1 we get

$$\begin{aligned}
c_1 \|C_g\| &\geq \left(\frac{1}{2}\right)^{\frac{2n-1}{p}} M_q\left(\frac{1+|w|}{2}, g f_w\right) \varphi_2(|w|) \left\{ \int_{\frac{1+|w|}{2}}^{\frac{3+|w|}{4}} \frac{(1-r)^{pb-1}}{(1-|w|)^{pb}} dr \right\}^{\frac{1}{p}} \\
&= \frac{1}{4^b} \left\{ \frac{2^{pb} - 1}{pb 2^{2n-1}} \right\}^{\frac{1}{p}} M_q\left(\frac{1+|w|}{2}, g f_w\right) \varphi_2(|w|) \\
&\geq \frac{1}{4^b} \left\{ \frac{2^{pb} - 1}{pb 2^{2n-1}} \right\}^{\frac{1}{p}} \frac{1}{\{4n \cdot 36^n\}^{1/q}} |g(w) f_w(w)| (1-|w|^2)^{\frac{n}{q}} \varphi_2(|w|) \\
&= \frac{1}{4^b} \left\{ \frac{2^{pb} - 1}{pb 2^{2n-1}} \right\}^{\frac{1}{p}} \frac{1}{\{4n \cdot 36^n\}^{1/q}} |g(w)| \frac{\varphi_2(|w|)}{\varphi_1(|w|)}.
\end{aligned}$$

This means that (1.1) holds.

If $p = \infty$, then

$$\begin{aligned}
c_1 \|C_g\| &\geq \|C_g\| \|f_w\|_{\infty, q, \varphi_1} \geq \|g f_w\|_{\infty, q, \varphi_2} \\
&= \sup_{0 \leq r < 1} M_q(r, g f_w) \varphi_2(r) \geq \sup_{\frac{|w|+1}{2} \leq r \leq \frac{3+|w|}{4}} M_q(r, g f_w) \varphi_2(r) \\
&\geq \frac{1}{4^b} M_q\left(\frac{|w|+1}{2}, g f_w\right) \varphi_2(|w|) \\
&\geq \frac{1}{4^b \{4n \cdot 36^n\}^{1/q}} |g(w) f_w(w)| (1-|w|^2)^{\frac{n}{q}} \varphi_2(|w|) \\
&= \frac{1}{4^b \{4n \cdot 36^n\}^{1/q}} |g(w)| \frac{\varphi_2(|w|)}{\varphi_1(|w|)}.
\end{aligned}$$

This shows that (1.1) holds. This proof is completed. \square

Proof of Theorem B. If $p = q$, then the result has been proved in Theorem A. So we discuss the case $p < q$ only.

If $g \in M(H(p, p, \varphi_1), H(q, q, \varphi_2))$, then $g \in H(B)$ is obvious. For any $w \in B$, we take

$$f_w(z) = \frac{(1 - |w|^2)^{b+1}}{\varphi_1(|w|)(1 - \langle z, w \rangle)^{n/p+b+1}}, \quad \text{where } b = \max\{b_1, b_2\}.$$

By Theorem A we know $\|f_w\|_{p,p,\varphi_1} \leq c$. By (2.1)–(2.5) we have

$$\begin{aligned} c\|C_g\| &\geq \|gf_w\|_{q,q,\varphi_2} = \left\{ \int_0^1 r^{2n-1} (1-r)^{-1} \varphi_2^q(r) M_q^q(r, gf_w) dr \right\}^{\frac{1}{q}} \\ &= \left(\frac{1}{2n} \right)^{\frac{1}{q}} \left\{ \int_B (1 - |z|)^{-1} \varphi_2^q(|z|) |g(z)|^q |f_w(z)|^q dv(z) \right\}^{\frac{1}{q}} \\ &\geq \left(\frac{1}{2n} \right)^{\frac{1}{q}} \left\{ \int_{E^*(w, \frac{1}{2})} (1 - |z|)^{-1} \varphi_2^q(|z|) |g(z)|^q |f_w(z)|^q dv(z) \right\}^{\frac{1}{q}} \\ &= \left(\frac{1}{2n} \right)^{\frac{1}{q}} \left\{ \int_{E^*(w, \frac{1}{2})} (1 - |z|)^{-1} \varphi_2^q(|z|) |g(z)|^q \right. \\ &\quad \times \left. \frac{(1 - |w|^2)^{q(b+1)}}{\varphi_1^q(|w|) |1 - \langle z, w \rangle|^{q(n/p+b+1)}} dv(z) \right\}^{\frac{1}{q}} \\ &\geq \left(\frac{1}{2n} \right)^{\frac{1}{q}} \left(\frac{1}{3} \right)^{\frac{1}{q}+b} \left(\frac{1}{2} \right)^{\frac{n}{p}+b+1} (1 - |w|^2)^{-\frac{1}{q}-\frac{n}{p}} \frac{\varphi_2(|w|)}{\varphi_1(|w|)} \\ &\quad \times \left\{ \int_{E^*(w, \frac{1}{2})} |g(z)|^q dv(z) \right\}^{\frac{1}{q}} \\ &\geq \left(\frac{1}{2n} \right)^{\frac{1}{q}} \left(\frac{1}{3} \right)^{\frac{n+2}{q}+b} \left(\frac{1}{2} \right)^{\frac{n}{p}+\frac{2n}{q}+b+1} (1 - |w|^2)^{(\frac{1}{q}-\frac{1}{p})n} \frac{\varphi_2(|w|)}{\varphi_1(|w|)} |g(w)|. \end{aligned}$$

This shows that (1.2) holds.

Conversely, if $g \in H(B)$ and (1.2) holds. For any $f \in H(p, p, \varphi_1)$, by Lemma 2.2 we have

$$\begin{aligned} \|gf\|_{q,q,\varphi_2} &= \left\{ \int_0^1 r^{2n-1} (1-r)^{-1} \varphi_2^q(r) M_q^q(r, gf) dr \right\}^{\frac{1}{q}} \\ &= \left(\frac{1}{2n} \right)^{\frac{1}{q}} \left\{ \int_B (1 - |z|)^{-1} \varphi_2^q(|z|) |g(z)|^q |f(z)|^q dv(z) \right\}^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
&\leq c \|f\|_{p,p,\varphi_1}^{1-\frac{p}{q}} \left\{ \int_B |f(z)|^p |g(z)|^q (1-|z|^2)^{(1-\frac{q}{p})n-1} \right. \\
&\quad \times \left. \varphi_2^q(|z|) \varphi_1^{p-q}(|z|) dv(z) \right\}^{\frac{1}{q}} \\
&\leq c \sup_{z \in B} \left\{ (1-|z|^2)^{(\frac{1}{q}-\frac{1}{p})n} \frac{\varphi_2(|z|)}{\varphi_1(|z|)} |g(z)| \right\} \|f\|_{p,p,\varphi_1}^{1-\frac{p}{q}} \\
&\quad \times \left\{ \int_B |f(z)|^p (1-|z|)^{-1} \varphi_1^p(|z|) dv(z) \right\}^{\frac{1}{q}} \leq c_1 \|f\|_{p,p,\varphi_1}.
\end{aligned}$$

This implies that $g \in M(H(p, p, \varphi_1), H(q, q, \varphi_2))$. \square

Proof of Theorem C. The sufficient condition is quite easy by Hölder inequality, and the details are omitted there.

Conversely, if $g \in M(H(p, p, \varphi_1), H(q, q, \varphi_2))$, then

$$\begin{aligned}
&\left\{ \int_B (1-|z|)^{-1} \varphi_2^q(|z|) |g(z)|^q |f(z)|^q dv(z) \right\}^{\frac{1}{q}} \\
&\leq (2n)^{\frac{1}{q}-\frac{1}{p}} \|C_g\| \left\{ \int_B (1-|z|)^{-1} \varphi_1^p(|z|) |f(z)|^p dv(z) \right\}^{\frac{1}{p}}
\end{aligned}$$

holds for all $f \in H(p, p, \varphi_1)$.

Let $r_0 = (e - e^{-1})/(e + e^{-1})$. We take $d\mu(z) = (1-|z|)^{-1} \varphi_2^q(|z|) |g(z)|^q dv(z)$ in Lemma 2.3. By (2.1), (2.4) and (2.5) we get

$$\begin{aligned}
\mu(E^*(z, r_0)) &= \int_{E^*(z, r_0)} (1-|w|)^{-1} \varphi_2^q(|w|) |g(w)|^q dv(w) \\
&\geq r_0^{2n} \left(\frac{1-r_0}{1+r_0} \right)^{n+2+qb} (1-|z|^2)^n \varphi_2^q(|z|) |g(z)|^q.
\end{aligned}$$

This means that

$$\hat{\mu}(|z|) \geq r_0^{2n} \left(\frac{1-r_0}{1+r_0} \right)^{n+2+qb} \frac{\varphi_2^q(|z|) |g(z)|^q}{\varphi_1^p(|z|)}. \quad (3.4)$$

Thus, by Lemma 2.3 we have

$$\begin{aligned}
\infty &> \int_B \hat{\mu}(|z|)^s (1-|z|)^{-1} \varphi_1^p(|z|) dv(z) \\
&\geq \left\{ r_0^{2n} \left(\frac{1-r_0}{1+r_0} \right)^{n+2+qb} \right\}^{\frac{p}{p-q}} \int_B (1-|z|)^{-1} \left[\frac{\varphi_2(|z|)}{\varphi_1(|z|)} \right]^k |g(z)|^k dv(z).
\end{aligned}$$

This shows that $g \in H(k, k, \varphi_2/\varphi_1)$. \square

Corollary 1. Suppose $0 < p < \infty$, $0 < q < \infty$, and φ is a normal function on $[0, 1)$. Then

- (1) $g \in M(L_a^p(\varphi), L_a^q(\varphi))$ if and only if $g \in H^\infty$ for $p = q$;
- (2) $g \in M(L_a^p(\varphi), L_a^q(\varphi))$ if and only if $g \equiv 0$ for $p \neq q$.

Proof. (1) If $p = q$, by Theorem A we can get the result.

(2) If $p < q$, by Theorem B and the maximum modulus principle we have $g \equiv 0$.

If $p > q$ and $g \in M(L_a^p(\varphi), L_a^q(\varphi))$, then, for any $0 < t < 1$, by Lemma 2.3 and (3.4) we have

$$\begin{aligned} \infty > N_0 &= \int_B \hat{\mu}(|z|)^s (1 - |z|)^{-1} \varphi^p(|z|) dv(z) \\ &\geq \left\{ r_0^{2n} \left(\frac{1 - r_0}{1 + r_0} \right)^{n+2+qb} \right\}^{\frac{p}{p-q}} \int_B (1 - |z|)^{-1} |g(z)|^{\frac{pq}{p-q}} dv(z) \\ &= 2n \left\{ r_0^{2n} \left(\frac{1 - r_0}{1 + r_0} \right)^{n+2+qb} \right\}^{\frac{p}{p-q}} \int_0^1 \frac{r^{2n-1}}{1 - r} M_{\frac{pq}{p-q}}^{\frac{pq}{p-q}}(r, g) dr \\ &\geq 2n \left\{ r_0^{2n} \left(\frac{1 - r_0}{1 + r_0} \right)^{n+2+qb} \right\}^{\frac{p}{p-q}} t^{2n-1} M_{\frac{pq}{p-q}}^{\frac{pq}{p-q}}(t, g) \int_t^{2t-t^2} \frac{1}{1 - r} dr. \end{aligned}$$

Thus

$$M_{\frac{pq}{p-q}}^{\frac{pq}{p-q}}(t, g) \leq \frac{ct^{1-2n}}{\log(1-t)^{-1}} \rightarrow 0 \quad \text{as } t \rightarrow 1^-.$$

This means $g \equiv 0$. \square

Corollary 2. Suppose $0 < p < \infty$, $0 < q < \infty$. Then

- (1) $g \in M(L_a^p, L_a^q)$ if and only if $g \in L_a^k$ for $p > q$, where $1/k = 1/q - 1/p$;
- (2) $g \in M(L_a^p, L_a^q)$ if and only if $g \in H^\infty$ for $p = q$;
- (3) $g \in M(L_a^p, L_a^q)$ if and only if $g \equiv 0$ for $p < q$.

Proof. (1) If $p > q$, by taking $\varphi_1(r) = (1 - r)^{1/p}$ and $\varphi_2(r) = (1 - r)^{1/q}$ in Theorem C we can obtain the result.

(2) If $p = q$, taking $\varphi_1(r) = \varphi_2(r) = (1 - r)^{1/p}$ in Theorem A.

(3) If $p < q$, by taking $\varphi_1(r) = (1 - r)^{1/p}$, $\varphi_2(r) = (1 - r)^{1/q}$ in Theorem B and using the maximum modulus principle we have $g \equiv 0$. \square

Corollary 3. Suppose $-\infty < \alpha < 0$, $-\infty < \beta < 0$. Then $g \in M(D_\alpha, D_\beta)$ if and only if $g \in H(B)$ and $|g(z)| = O((1 - |z|)^{(\beta-\alpha)/2})$.

Proof. Since $D_\alpha = H(2, 2, (1 - r)^{-\alpha/2})$ and $D_\beta = H(2, 2, (1 - r)^{-\beta/2})$, we can get the result by Theorem A. \square

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